NOTE ON CAUCHY'S NUMBERS.

By Prof. A. S. Chessin, Baltimore, Md.

Let p be any integer *positive, negative, or zero ;* q and j any *positive integers or zero ;* develop the expression

$$x^{-p}\left[x+\frac{1}{x}\right]^{j}\left[x-\frac{1}{x}\right]^{q} \tag{A}$$

in powers of x and denote the constant term of this development by $N_{-p,j,q}$. The number $N_{-p,j,q}$ is called a *number of Cauchy.* These numbers play an important role in certain developments used in Celestial Mechanics. In this note a formula will be given by which Cauchy's numbers can be directly and easily calculated.

First, let us recall some important properties of these numbers ; namely, that

when $-p+j+q=0$;
$$N_{-p,j,q}=1\,, \tag{1}$$
$$N_{-p,j,q}=0\,, \tag{2}$$

when $-p+j+q$ is a negative number or when it is odd ;

$$N_{p,j,q}=(-1)^{q}\,N_{-p,j,q}\,; \tag{3}$$

$$N_{-p,j+1,q}=N_{-p+1,j,q}+N_{-p-1,j,q}\,; \tag{4}$$

$$N_{-p,j,q+1}=N_{-p+1,j,q}-N_{-p-1,j,q}\,. \tag{5}$$

From the last two formulas, by a successive application of the same, the following are easily obtained :

$$\begin{aligned}N_{-p,j,q}=N_{-p+m,j-m,q}+\left[\frac{m}{1}\right]N_{-p+m-2,j-m,q}+\left[\frac{m}{2}\right]N_{-p+m-4,j-m,q}\\+\left[\frac{m}{3}\right]N_{-p+m-6,j-m,q}+\cdots+\left[\frac{m}{1}\right]N_{-p-m+2,j-m,q}+N_{-p-m,j-m,q}\,,\end{aligned} \tag{6}$$

$$\begin{aligned}N_{-p,j,q}=N_{-p+m,j,q-m}-\left[\frac{m}{1}\right]N_{-p+m-2,j,q-m}+\left[\frac{m}{2}\right]N_{-p+m-4,j,q-m}\\-\left[\frac{m}{3}\right]N_{-p+m-6,j,q-m}+\cdots+(-1)^{m}\,N_{-p-m,j,q-m}\,.\end{aligned} \tag{7}$$

In particular, if $m = j$, formula (6) gives

$$N_{-p,j,q} = N_{-p+j,0,q} + \begin{bmatrix} j \\ 1 \end{bmatrix} N_{-p+j-2,0,q} + \begin{bmatrix} j \\ 2 \end{bmatrix} N_{-p+j-4,0,q} + \begin{bmatrix} j \\ 3 \end{bmatrix} N_{-p+j-6,0,q} + \ldots + N_{-p-j,0,q}\,; \tag{6'}$$

and if $m = q$, formula (7) gives

$$N_{-p,j,q} = N_{-p+q,j,0} - \begin{bmatrix} q \\ 1 \end{bmatrix} N_{-p+q-2,j,0} + \begin{bmatrix} q \\ 2 \end{bmatrix} N_{-p+q-4,j,0} - \begin{bmatrix} q \\ 3 \end{bmatrix} N_{-p+q-6,j,0} + \ldots + (-1)^q N_{-p-q,j,0}\,. \tag{7'}$$

Combining formulas (6′) and (7′) together, we easily obtain the following:

$$\begin{aligned} N_{-p,j,q} &= N_{-p+j+q,0,0} + (j-q)\, N_{-p+j+q-2,0,0} \\ &+ \left[\begin{bmatrix} j \\ 2 \end{bmatrix} - \begin{bmatrix} j \\ 1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} q \\ 2 \end{bmatrix} \right] N_{-p+j+q-4,0,0} + \ldots \\ &+ \left[\begin{bmatrix} j \\ n \end{bmatrix} - \begin{bmatrix} j \\ n-1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} j \\ n-2 \end{bmatrix} \begin{bmatrix} q \\ 2 \end{bmatrix} - \ldots + (-1)^n \begin{bmatrix} q \\ n \end{bmatrix} \right] N_{-p+j+q-2n,0,0} \\ &+ \ldots + (-1)^q N_{-p-j-q,0,0}\,. \end{aligned} \tag{8}$$

But

$$N_{-p+j+q-2n,0,0} = 0\,, \text{ if } -p+j+q-2n \gtrless 0\,,$$

$$N_{-p+j+q-2n,0,0} = 1\,, \text{ if } -p+j+q-2n = 0\,.$$

In fact, if $j = q = 0$ in formula (A), the constant part in the development is zero, unless $p = 0$, in which case it is unity. Hence

$$N_{-p,j,q} = \begin{bmatrix} j \\ n \end{bmatrix} - \begin{bmatrix} j \\ n-1 \end{bmatrix} \begin{bmatrix} q \\ 1 \end{bmatrix} + \begin{bmatrix} j \\ n-2 \end{bmatrix} \begin{bmatrix} q \\ 2 \end{bmatrix} - \ldots + (-1)^n \begin{bmatrix} q \\ n \end{bmatrix}$$

$$n = \tfrac{1}{2}(-p+j+q)\,.$$

This formula gives the solution of the proposed problem.

JOHNS HOPKINS UNIVERSITY, *October*, 1895.